Math 201
Preparation for Quiz #2

I. Binomial theorem

Example 1 *Write down the first four terms in the Maclaurin series expansion of* $f(x) = \sqrt[3]{1+2x}$.

Solution 2

$$f(x) = (1+2x)^{1/3} = 1 + \frac{1}{3}(2x) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}(2x)^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}(2x)^3 + \cdots$$

$$= 1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}(2x)^3 + \cdots.$$

II. Remainders

Example 3 *Estimate the remainder (that is give an inequality) after 10 terms of each of the following series*

$$(a)\sum_{n=1}^{\infty}\frac{1}{n^{1.7}};\,(b)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{0.6}}.$$

Solution 4 *(a) In the series* $\sum_{n=1}^{\infty}\frac{1}{n^{1.7}}$, *the remainder after* 10 *terms is* $R_{10} = \sum_{n=11}^{\infty}\frac{1}{n^{1.7}}$. *Since the function* $\frac{1}{x^{1.7}}$ *is positive and decreasing on* $[1,\infty)$ *we have*

$$R_{10} = \sum_{n=11}^{\infty}\frac{1}{n^{1.7}} < \int_{10}^{\infty}\frac{1}{x^{1.7}}dx = \int_{10}^{\infty}x^{-1.7}dx = \frac{x^{-0.7}}{-0.7}|_{10}^{\infty} = \frac{(10)^{-0.7}}{0.7} \approx 0.28504.$$

(b) The series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{0.6}}$ is strictly alternating and the n^{th} term decreases in absolute value and tends to 0, therefore the remainder after 10 terms $R_{10} = \sum_{n=11}^{\infty}\frac{(-1)^{n+1}}{n^{0.6}}$, has the same sign as $\frac{(-1)^{12}}{(12)^{0.6}}$ and is less than $\frac{1}{(12)^{0.6}} \approx 0.225\,16$. This means that if we approximate the full infinite sum with just the sum of the first 10 terms, we would be ove estimating it by about 0.225 16.

Example 5 *For what values of* x *can we replace* $\cos x$ *by* $1 - \frac{x^2}{2!}$ *with an error of magnitude no greater than* 3×10^{-4}

Solution 6 *According to the Alternating Series Estimation Theorem (section 10.6), the error in truncating the series for* $\cos x$ *which is*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

after $\frac{x^2}{2!}$, is (for small values of x), no greater than

$$\left|\frac{x^4}{4!}\right| = \frac{|x|^4}{120}.$$

So all we need is to find x so that this is less than 3×10^{-4} *. So we solve the inequality*

$$\frac{|x|^4}{120} < 3 \times 10^{-4}$$

and we obtain

$$|x| < \sqrt[4]{360 \times 10^{-4}} = 0.43559.$$

III. Polar coordinates

1. **Example 7** *Sketch the two curves given in polar coordinates by the equations $r = 3(1 - \sin\theta)$, and $r = 3\sin\theta$ and find all their points of intersection..*

IV. Partial derivatives

Example 8 *Does $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + \sin^2 y}$ exist? Justify your answer.*

Example 9 *Let $f(x,y) = x^2 y + e^x \cos y$. Compute ∇f and Δf.*

Example 10 *Find the domain of definition of the function $f(x,y) = \sqrt{4 - x - x^2}$. Also describe the level curves of this function.*

Example 11 *GIve an example of a function which possesses partial derivatives but is not continuous at $(0,0)$. Prove your answer (don't just give the example).*

Example 12 *Suppose that f is a differentiable function of two variables and $w = f(ts^2, \frac{s}{t})$, $\frac{\partial f}{\partial x}(x,y) = xy$, $\frac{\partial f}{\partial y}(x,y) = \frac{x^2}{2}$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$.*

Example 13 *Problem 44 page 783 in textbook.*

Example 14 *The derivative of a function $f(x,y)$ at $P_0(1,2)$ in the direction of $\mathbf{i} + \mathbf{j}$ is $2\sqrt{2}$, and in the direction $-2\mathbf{j}$ is -3. What is the derivative in the direction of $-\mathbf{i} - 2\mathbf{j}$? Justify your answer.*

Example 15 *Find an equation of the plane tangent to the surface $x^2 + y^2 - 4y = 0$ at the point $P(2, 2, \sqrt{8})$. Then find equations for the line of intersection of this plane with the xy plane.*

Example 16 *The directional derivative of $f(x,y,z)$ at a point P is greatest in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. In this direction, the value of the derivative is $2\sqrt{3}$. Find $\boldsymbol{\nabla} f(P)$, and then find the directional derivative of f at P in the direction of $\mathbf{i} + \mathbf{j}$.*